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RANK ONE LIGHTLY MIXING

BY

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ABSTRACT

A rank one transformation T was constructed by Chacón that is weakly mixing but not mixing. We will show that T is lightly mixing, not partially mixing, and not lightly 2-mixing.

1. Introduction

For convenience, a transformation that is lightly mixing but not partially mixing (see §2 for definitions) will be called *just lightly mixing*. The first example of a transformation that is just lightly mixing was constructed in [BCQ], where it is called sequence mixing. Higher order lightly mixing was considered in [FT]. The examples of transformations that are just lightly mixing in [BCQ] and [FT] were obtained as countable products of partially mixing transformations [FO1, FO2, F2]. A lingering question was whether just lightly mixing transformations could be constructed directly by cutting and stacking. In particular, it was asked in [KI] if there is a rank one construction for just lightly mixing. In [KI] it was shown that countable products of lightly mixing transformations are lightly mixing.

Our purpose is to show that an example of a rank one transformation in [C] is just lightly mixing. This example is different from Chacón's transformation [F1, p. 86] which is not lightly mixing (see §4). The example in [C] is obtained simply by cutting in half and stacking with a single spacer added at each stage. We also show that this transformation is not lightly 2-mixing. The only other examples of transformations that are just lightly mixing but not lightly 2-mixing were obtained as countable products of transformations that are partially mixing but not lightly 2-mixing [F2, FT].

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In [KA] Kalikow proved that a rank one transformation that is mixing is also 2-mixing (also called three-fold mixing). The example discussed below shows that the analogue of this result does not carry over to lightly mixing.

2. Preliminaries

Let $(X,\mathcal{B},\mu)$ be a measure space isomorphic to the unit interval with Lebesgue measure and let T be an invertible measure preserving map of X onto X. T is *lightly mixing* if for all A and B of positive measure we have

$$\liminf_{n\to\infty} \mu(T^nA \cap B) > 0. \tag{2.1}$$

T is *lightly* 2-*mixing* if for all A, B, and C of positive measure we have

$$\liminf_{n,m\to\infty} \mu(T^n(T^mA \cap B) \cap C) > 0. \tag{2.2}$$

T is *partially mixing* if there exists $\alpha > 0$ such that

$$\liminf_{n\to\infty} \mu(T^nA \cap B) \geq \alpha\mu(A)\mu(B), \qquad \text{for all } A,B \in \mathcal{B}. \tag{2.3}$$

In [KI] it was pointed out that it suffices to verify (2.1) with $A = B$. We will use the following slightly weaker characterization of lightly mixing.

(2.4) Lemma. *T is lightly mixing if and only if for each set A of positive measure there exists N such that $\mu(T^nA \cap A) > 0$ for all $n \geq N$.*

Proof. Suppose T is not lightly mixing; hence there exists E of positive measure such that $\liminf_{n\to\infty} \mu(T^nE \cap E) = 0$. Choose $n_k \to \infty$ such that $E_k = T^{n_k}E \cap E$ satisfies $\mu(E_k) < \mu(E)/3^k$. Let $A = E - \bigcup_{k=1}^{\infty} E_k$. Then

$$\mu(A) \geq \mu(E) - \sum_{k=1}^{\infty} \mu(E_k) > \mu(E) - \mu(E)/2 = \mu(E)/2.$$

The intersection $T^{n_k}A \cap A$ is empty for all k. Contradiction. ■

3. Example

The construction of the rank one transformation T in [C] is most conveniently described in terms of the n-blocks B_n for $n = 1,2,3,\ldots$. Let $B_1 = (0)$ and let s denote a spacer. By induction, we define $B_{n+1} = B_n B_n s$. If h_n is the length of B_n, then $h_{n+1} = 2h_n + 1$.

In terms of cutting and stacking, let C_n denote the single column of height h_n corresponding to B_n. Therefore C_{n+1} is obtained by cutting C_n in half and stacking the right half above the left half with an additional spacer level denoted by S_{n+1} placed on top. We can begin with $C_1 = ([0, 1/2))$ and let $S_{n+1} = [1 - 1/2^n, 1 - 1/2^{n+1})$ for all $n \geq 1$. Thus we obtain $T = \lim_{n\to\infty} T_{C_n}$ defined on $[0,1)$. In Fig. 3.1 we show C_n of height h_n with top level S_n. The arrows show the action of T.

Let $I_{n,i}$ denote the ith level of C_n starting at the top for $1 \leq i \leq h_n$, as in (3.1). The construction implies that $T^{h_n} S_n$ is the union of the spacer interval S_{n+h_n} and the h_n many intervals $T^{h_n} S_n \cap I_{n,i}$ for $1 \leq i \leq h_n$. These latter are indicated by bold lines in (3.1). The interval lengths decrease by a factor of 1/2 and we have $\mu(T^{h_n} S_n \cap I_{n,i}) = \mu(S_n)/2^i$. We will refer to the configuration of these intervals as in Fig. 3.1 as a *crescent*.

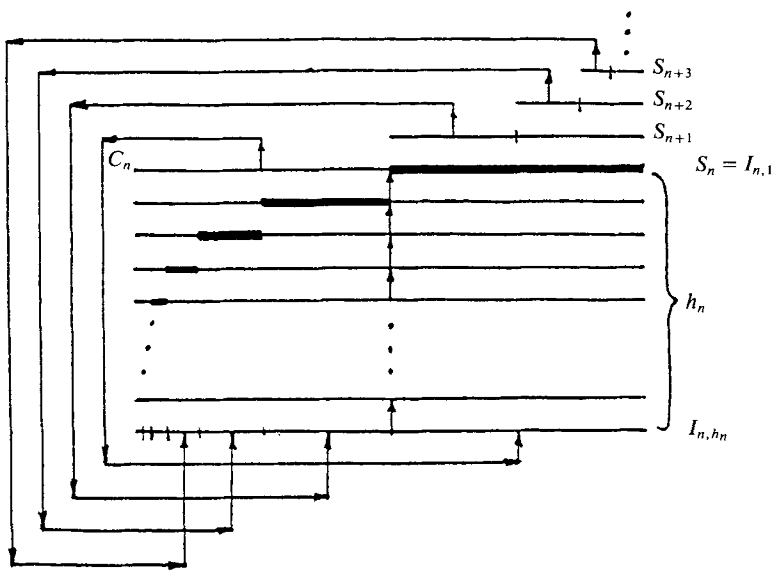

Fig. 3.1.

Fix k and let $n > k$. The column C_k appears in C_n as 2^{n-k} disjoint groups of h_k consecutive levels of C_n. Each of these groups of h_k consecutive levels will be called a *copy* of C_k, or a *C_k-copy*. Thus C_k appears in C_n as 2^{n-k} disjoint copies of C_k, as indicated in Fig. 3.2.

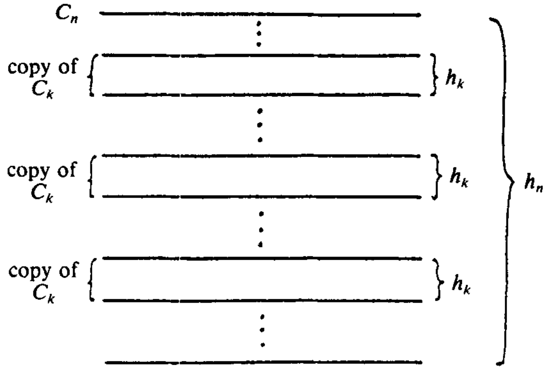

Fig. 3.2.

For future reference, consider how many spacers separate a C_k-copy in C_n from the C_k-copy below it. If $n = k + 1$, then one of the two copies of C_k in C_{k+1} is preceded by $u = 0$ spacers. If $n = k + 2$, then two of the four C_k-copies in C_{k+2} are preceded by $u = 0$ spacers and one C_k-copy is preceded by $u = 1$ spacers. Proceeding inductively we can show, of the 2^{n-k} copies of C_k in C_n, that $2^{n-k}/2^{u+1}$ copies are preceded by u spacers, for $0 \leq u < n - k$. Thus the fraction of copies of C_k in C_n that are preceded by *at most* u spacers is

$$1/2 + 1/4 + \cdots + 1/2^{u+1} = 1 - 1/2^{u+1}. \tag{3.3}$$

Now let I be any level in C_n and consider the set $T^{h_n}I$. This set forms a crescent of intervals on the levels of C_n starting at I and below, where the interval lengths decrease geometrically by a factor of 1/2. These intervals are indicated by bold lines in Fig. 3.4 and extend to the bottom of C_n. The remainder of $T^{h_n}I$ is the spacer interval S_{n+h_n-i+1}, if I was the ith level from the top in C_n.

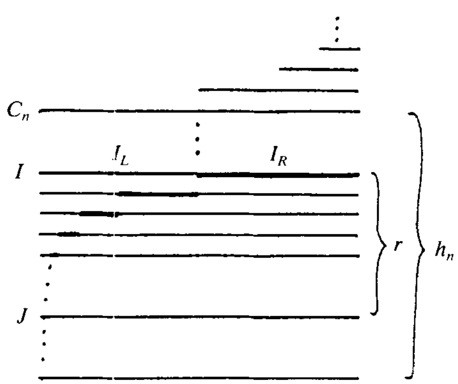

Fig. 3.4.

In particular, if J is a level in C_n located $r-1$ levels below I as in (3.4), then

(3.5) *$T^{h_n}I \cap J$ is a subset of: The leftmost subinterval of J with length $\mu(J)/2^r$.*

We also note that if I_L is the left half of I, then $T^{h_n}I_L = I_R$, where I_R is the right half of I in (3.4). The set $T^{h_n}I_R$ is the union of the intervals in the crescent on the levels below I and the single spacer interval S_{n+h_n-i+1}. Thus $T^{h_n}I_R$ is contained in the left half of C_n, except for the single spacer interval. These properties of a crescent are the key ingredients used to verify that T satisfies the conditions stated below.

THEOREM 3.6. *The transformation T satisfies the following conditions.*
(a) *T is not partially mixing.*
(b) *T is lightly mixing.*
(c) *T is not lightly 2-mixing.*

PROOF OF (a). Let $\alpha > 0$ and fix k so large that $h_k/2^{h_k} < \alpha/4$. Let $A = S_k$ be the top level of C_k and let B be the bottom level in C_k; hence B is $h_k - 1$ levels below A. Now for $n > k$ the sets A and B appear as levels $I_j \subset A$ and $J_j \subset B$, for $1 \le j \le 2^{n-k}$, in the 2^{n-k} copies of C_k in C_n. The level I_j is $h_k - 1$ levels above J_j. For an I_i which is below J_j the intersection $T^{h_n}I_i \cap J_j$ is empty. For those I_i above J_j each intersection $T^{h_n}I_i \cap J_j$ is a subinterval of the left-most interval of J_j of length $\mu(J_j)/2^{h_k}$, as (3.5) asserts. As i varies these subintervals are disjoint. So

$$\mu(T^{h_n}A \cap B) = \sum_{j=1}^{2^{n-k}} \mu(T^{h_n}A \cap J_j) \le \sum_{j=1}^{2^{n-k}} 2^{-h_k}\mu(J_j) = 2^{-h_k}\mu(B).$$

Since $\mu(A) \approx 1/h_k$ we see that $\mu(T^{h_n}A \cap B) < 2^{-h_k}(2h_k \cdot \mu(A))\mu(B)$. Hence $\mu(T^{h_n}A \cap B) < (\alpha/2)\mu(A)\mu(B)$. Sending $n \to \infty$ shows that T is not α-mixing. ■

PROOF OF (b). Fix a set A of positive measure. We need to find M such that

$$\mu(T^m A \cap A) > 0, \qquad \text{for all } m \ge M. \tag{3.7}$$

First choose k so large that $\mu(C_k) > 0.9$ and so that some C_k-level I has $\mu(I \cap A)$ exceeding $0.99\mu(I)$. There is no harm in moving A by a power of T, and then shrinking A. So we may assume that $A \subset S_k$, the top level of column C_k, and that $\mu(A)/\mu(S_k) > (0.99)$.

Fix a small $\delta > 0$ and large $N > k$ to be specified later. For $n \ge N$, the set S_k appears as 2^{n-k} C_n-levels I_i for $1 \le i \le 2^{n-k}$; the top level in each of the 2^{n-k} copies of C_k in C_n. Say that a level I of C_n is *accurate* if

$$\mu(I \cap A) > (1-\delta)\mu(I).$$

Say that a C_k-copy in C_n is *good* if: Its top level is accurate; the top level of the C_k-copy below it is accurate; and at most $u = 10$ spacers separate these two C_k-copies. Now with N sufficiently large we can ask that

$$\#[\text{accurate } C_n\text{-levels } I] > \left(\frac{\mu(A)}{\mu(S_k)} - 0.01\right)\cdot 2^{n-k} \overset{\text{note}}{>} 0.98\cdot 2^{n-k}, \tag{3.8}$$

since A becomes progressively more measurable with respect to the C_n-levels as n increases. Thus 0.02 dominates the fraction of C_k-copies whose top level is inaccurate. Together with (3.3) this says

$$\mu(G_n) > (1 - 2\cdot(0.02) - 2^{-11})\mu(C_k) > 0.8, \tag{3.8'}$$

where G_n denotes the union of all good C_k-copies in C_n.

The computation. We establish (3.7) for M set equal to h_N. For an $m \geq M$ let $n \geq N$ denote the value for which

$$h_n \leq m < h_{n+1} \overset{\text{note}}{=} 2h_n + 1.$$

Let $S_{n,k} = \bigcup_{i=1}^{h_k} S_{n+i}$, as in (3.1). Suppose I is an accurate level in C_n whose image $T^m I$ is disjoint from $S_{n,k}$. Then $T^m I$ will appear in C_n as a single or double crescent, as indicated by the bold lines in Fig. 3.9. When $m = h_n$, the set $T^m I$ will appear as the crescent in Fig. 3.9a. As m increases, the set moves upward. After the top right interval of $T^m I$ has filtered through $S_{n,k}$, the set $T^m I$ will appear as the pair of crescents in Fig. 3.9b.

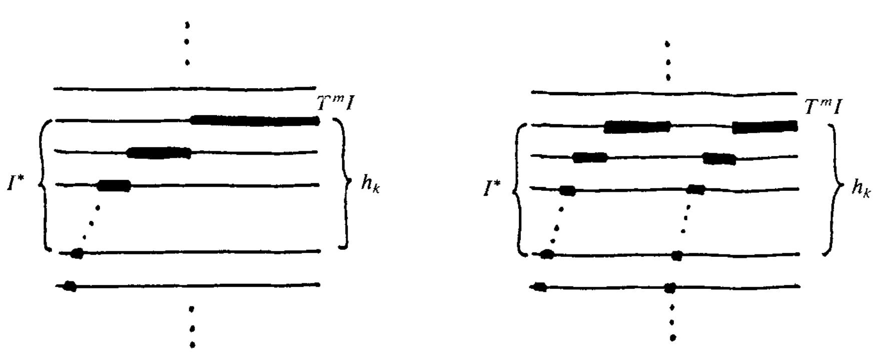

Fig. 3.9a. Fig. 3.9b.

Let I^* be the union of the h_k many C_n-levels that contain the top h_k levels of $T^m I$. Since I^* consists of h_k many levels in C_n we have that $\mu(I^*)$ equals the measure of a C_k-copy in C_n, which is $\mu(C_k)/2^{n-k}$. Let G_n^* denote the union of all such I^*. Since there is at most one I whose image fails to be disjoint from $S_{n,k}$,

$$\mu(G_n^*) > (\#\text{accurate } C_n\text{-levels} - 1)\frac{\mu(C_k)}{2^{n-k}}.$$

We may thus assume, by (3.8), that $\mu(G_n^*) > 0.98\mu(C_k) > (0.8)$.

Two cases. Since $\mu(G_n^* \cap G_n)$ is positive, there is an I^* and a good C_k-copy, call it D, which intersect in positive measure. Either (i) the top of I^* is below the top level of D or (ii) the top of I^* is above or coincides with the top of D.

For case (i), let J denote the top level of the C_k-copy below D. for case (ii), let J denote the top level of D itself. In either case, J is fewer than $h_k + 10$ levels below the top level of I^*. This is because there are at most 10 spacers between D and the C_k-copy below it. Since in either case both J and I are accurate levels of equal measure,

$$\begin{aligned}\mu(T^m(A) \cap A) &\geq \mu(T^m(A \cap I) \cap (A \cap J)) \\ &\geq \mu(T^m I \cap J) - \delta\mu(I) - \delta\mu(J) \\ &\geq \mu(J) \cdot (1/2^{h_k+10} - 2\delta).\end{aligned}$$

This latter will be positive, if we chose δ to be sufficiently small. Condition (3.7) is the consequence, as desired. ■

Proof of (c). Let $A = S_2$. Since h_2 (= 3) exceeds 1, we have that $(T^{-1}A) \cap A$ is empty. Now fix $n \geq 2$. Then A appears as a union $A = \bigcup I$ of certain levels I in C_n. Let $I = I_L \cup I_R$, where I_L (which depends on n) is the left half of I and I_R is the right half of I as in Fig. 3.4. We have $T^{h_n} I_L = I_R$ for each level I. Therefore $(T^{h_n-1} I_L) \cap A = \varnothing$ for each I since $(T^{-1}A) \cap A = \varnothing$. The construction implies that $(T^{h_n-1} I_R) \cap A \subset \bigcup I_L$; hence

$$\begin{aligned}(T^{h_n-1}A) \cap A &= T^{h_n-1}(\bigcup (I_L \cup I_R)) \cap A \\ &= \bigcup ((T^{h_n-1} I_R) \cap A) \subset \bigcup I_L.\end{aligned}$$

Now $T^{h_n} I_L = I_R \subset A$ implies $(T^{h_n} I_L) \cap A^c = \varnothing$. Hence

$$T^{h_n}((T^{h_n-1}A) \cap A) \cap A^c = \varnothing.$$

Sending $n \to \infty$ shows T to be not lightly 2-mixing. ■

4. Mixing conditions in rank one

Five successively stronger mixing conditions are (1) weakly mixing, (2) mildly mixing, (3) lightly mixing, (4) partially mixing, (5) mixing. A transformation is *mildly mixing* if (2.1) holds for all pairs $B = A^c$, whenever $0 < \mu(A) < 1$. Within rank one there are transformations having property (i) but not $(i+1)$, for $i = 1,2,3,4$. A weakly mixing transformation that is not mildly mixing is obtained by introducing rigidity in a weakly mixing rank one construction. This can be done by cutting into many columns and stacking with no spacer [F1, p. 135]. Chacón's transformation [F1, p. 86] is mildly mixing since it is prime and has trivial centralizer [J]. However, it is not lightly mixing since if I is the top of the 2-stack, then $T^{h_n}I \subset (I \cup T^{-1}I)$, where h_n is the height of the n-stack. Therefore $T^{h_n}I \cap (I \cup T^{-1}I)^c = \emptyset$. The example of this article distinguishes lightly mixing from partially mixing. Finally, Ornstein's rank one mixing construction [O] can be modified to obtain rank one partially mixing transformations as in (2.3) that are not mixing. The idea is to add the random spacers on the first α-fraction of the n-blocks in the $(n+1)$-block and no spacer between the remaining n-blocks in the $(n+1)$-block.

References

[BCQ] J. R. Blum, S. L. M. Christianson and D. Quiring, *Sequence mixing and α-mixing*, Illinois J. Math. **18** (1974), 131–135.

[C] R. V. Chacón, *Weakly mixing transformations which are not strongly mixing*, Proc. Am. Math. Soc. **22** (1969), 559–562.

[F1] N. A. Friedman, *Introduction to Ergodic Theory*, Van Nostrand Reinhold, New York, 1970.

[F2] N. A. Friedman, *Higher order partial mixing*, Contemporary Math. **26** (1984), 111–130.

[FO1] N. A. Friedman and D. S. Ornstein, *On partial mixing transformations*, Indiana Univ. Math. J. **20** (1971), 767–775.

[FO2] N. A. Friedman and D. S. Ornstein, *On mixing and partial mixing*, Illinois J. Math. **16** (1972), 61–68.

[FT] N. A. Friedman and E. Thomas, *Higher order sweeping out*, Illinois J. Math. **29** (1985), 401–417.

[J] A. del Junco, *A simple measure preserving transformation with trivial centralizer*, Pacific J. Math. **79** (1978), 357–362.

[KA] S. Kalikow, *Twofold mixing implies threefold mixing for rank one transformations*, Ergodic Theory and Dynamical Systems **4** (1984), 237–259.

[KI] J. King, *Lightly mixing is closed under countable products*, Isr. J. Math. **62** (1988), 341–346.

[O] D. S. Ornstein, *On the root problem in ergodic theory*, Proc. Sixth Berkeley Symp. Math. Stat. Prob. (1970), 347–356.